ELSEVIER Journal of Geometry and Physics 20 (1996) 404–406

JOURNAL OF GEOMETRY AND PHYSICS

Brief communication

A note on the Wodzicki residue

Thomas Ackermann [1]

Wasserwerkstr. 37, D-68309 Mannheim, Germany

Received 11 September 1995; revised 21 November 1995

Abstract

In this note we explain the relationship of the Wodzicki residue of (certain powers of) an elliptic differential operator P acting on sections of a complex vector bundle E over a closed compact manifold M and the asymptotic expansion of the trace of the corresponding heat operator e^{-tP}. In the special case of a generalized laplacian Δ and $\dim M > 2$, we thereby obtain a simple proof of the fact already shown in [KW], that the Wodzicki residue $\mathrm{res}(\Delta^{-n/2+1})$ is the integral of the second coefficient of the heat kernel expansion of Δ up to a proportional factor.

Subj. Class.: Non-commutative geometry
1991 MSC: 47F05, 53B50, 58G11
Keywords: Non-commutative residue; Heat kernel expansion

1. Introduction

In [KW] it was shown that the Wodzicki residue $\mathrm{res}(\Delta^{-n/2+1})$ of a generalized laplacian Δ on a complex vector bundle E over a closed compact manifold M, $\dim M = n > 2$, is the integral of the second coefficient of the heat kernel expansion of Δ up to a proportional factor. This observation reflects a more general property which we explain in this note, namely, the Wodzicki residue of (certain powers of) an elliptic differential operator P on E can be related with the coefficients in the asymptotic expansion of the trace of the corresponding heat operator e^{-tP}. Although this might be well known to mathematicians working in the field – our simple proof relies on two results made by Wodzicki [W] and Gilkey [Gi] – we think it is worth to restate it because of the growing importance of the Wodzicki residue for non-commutative geometry, for example, to incorporate gravity in the Connes–Lott model (cf. [CFF]). By the way we correct a misprint in [KW].

[1] E-mail: ackerm@euler.math.uni-mannheim.de.

SSDI 0393-0440(95)00061-5

2. Computing $\mathrm{res}(P^{-((n-k)/d)})$

Let E be a finite-dimensional complex vector bundle over a closed compact manifold M of dimension n. Recall that the non-commutative residue of a pseudo-differential operator $P \in \Psi\mathrm{DO}(E)$ can be defined by

$$\mathrm{res}(P) := (2\pi)^{-n} \int_{S^*M} \mathrm{tr}(\sigma^P_{-n}(x,\xi))\,\mathrm{d}x\,\mathrm{d}\xi, \tag{2.1}$$

where $S^*M \subset T^*M$ denotes the co-sphere bundle on M and σ^P_{-n} is the component of order $-n$ of the complete symbol $\sigma^P := \sum_i \sigma^P_i$ of P, cf. [W]. In his thesis, Wodzicki has shown that the linear functional $\mathrm{res} : \Psi\mathrm{DO}(E) \to \mathbb{C}$ is in fact the unique trace (up to multiplication by constants) on the algebra of pseudo-differential operators $\Psi\mathrm{DO}(E)$.

Now let $P \in \Psi\mathrm{DO}(E)$ be elliptic with $\mathrm{ord}\, P = d > 0$. It is well known (cf. [Gi]) that its zeta function $\zeta(P,s)$ is a holomorphic function on the half-plane $\mathrm{Re}\, s > n/d$ which continues analytically to a meromorphic function on $\mathbb{C}$ with simple poles at $\{(n-k)/d \mid k \in \mathbb{N} \setminus \{n\}\}$. For $n - k > 0$ with $k \in \mathbb{N}$ we have the following identities:

$$\mathrm{res}(P^{-((n-k)/d)}) = d \cdot \mathrm{Res}_{s=(n-k)/d}\,\zeta(P,s), \tag{2.2}$$

$$\mathrm{Res}_{s=(n-k)/d}\,\zeta(P,s) = a_k(P) \cdot \Gamma\left(\frac{n-k}{d}\right)^{-1}. \tag{2.3}$$

Here Γ is the gamma function and $a_k(P)$ denotes the coefficient of $t^{(k-n)/d}$ in the asymptotic expansion of $Tr_{L^2}\mathrm{e}^{-tP}$. The first equality was shown in [W], whereas (2.3) is a consequence of the Mellin transform which relates the zeta function and the heat equation and was already proven by Gilkey in [Gi]. Consequently we obtain [2]

$$a_k(P) = d^{-1} \cdot \Gamma\left(\frac{n-k}{d}\right) \cdot \mathrm{res}(P^{-((n-k)/d)}). \tag{2.4}$$

Now suppose $P = \Delta$ is a generalized laplacian and $k = 2$. Then $a_2(\Delta) = (4\pi)^{-n/2}\phi_2(\Delta)$ where $\phi_2(\Delta)$ denotes the integral over the diagonal part of the second coefficient of the heat kernel expansion of Δ. By using the well-known identity $\Gamma(z+1) = z\Gamma(z)$ it is easily verified that

$$(n-2)\phi_2(\Delta) = (4\pi)^{n/2} \cdot \Gamma\left(\frac{n}{2}\right) \cdot \mathrm{res}(\Delta^{-n/2+1}) \tag{2.5}$$

for $n > 2$. Note, that in the above-mentioned reference [KW] the Wodzicki residue was defined by

$$\widetilde{\mathrm{res}}(P) := \frac{\Gamma(n/2)}{2\pi^{n/2}} \int_{S^*M} \mathrm{tr}(\sigma^P_{-n}(x,\xi))\,\mathrm{d}x\,\mathrm{d}\xi,$$

[2] After completion of this work we have been told that the result (2.4) was independently recognized by Walze [Wa].

using $\mathrm{vol}(S^{n-1})^{-1} = \Gamma(n/2)/2\pi^{n/2}$ as a normalizing factor. Thus, Eq. (2.5) can be equivalently expressed as $\widetilde{\mathrm{res}}(\Delta^{-n/2+1}) = \frac{1}{2}(n-2)\phi_2(\Delta)$. The proportionality factor in their equation (4.10) contains therefore a misprint.

We conclude in exploiting the result that we have in hand in the case of a Clifford module $E = \mathcal{E}$ over a four-dimensional closed compact Riemannian manifold M and Δ being the square of a Dirac opertor D defined by a Clifford connection. Then Eq. (2.5) together with the well-known observation $\phi_2(D^2) \sim \int_M *r_M$, where $*$ is the Hodge star corresponding to the Riemannian metric and r_M denotes the scalar curvature of M, yields $\mathrm{res}(D^{-2}) \sim \int_M *r_M$. Thus, by the above derivation of (2.5), the fact – first announced by Connes [C] and shown by Kastler [K] using the symbol calculus – that the non-commutative residue of the inverse square of the Dirac operator is proportional to the Einstein–Hilbert action of general relativity, is almost obvious.

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